2.2 – Evaluating Determinants by Row Reduction

Theorem 2.2.1 Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Theorem 2.2.2 Let A be a square matrix. Then $\det(A) = \det\left(A^T\right)$.

Theorem 2.2.3 Elementary Row Operations
Let A be an $n \times n$ matrix.
a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$.
b) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.
c) If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then
$\det(B) = \det(A)$.

Pf (a): Let B be the matrix that results from multiplying the i^{th} row of A by k.
Expanding along the i^{th} row of A and B:

$$\det(A) = \sum_{l=1}^{n} a_{il} C_{il} \quad \leftarrow \text{cofactor}$$

$$\det(B) = \sum_{l=1}^{n} k a_{il} C_{il} = k \sum_{l=1}^{n} a_{il} C_{il} = k\det(A)$$

Consider $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{vmatrix}$ and $\begin{vmatrix} 10 & 15 & 20 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{vmatrix}$

$= 2\begin{vmatrix} \; \end{vmatrix} - 3\begin{vmatrix} \; \end{vmatrix} + 4\begin{vmatrix} \; \end{vmatrix}$ $\quad = 10\begin{vmatrix} \; \end{vmatrix} - 15\begin{vmatrix} \; \end{vmatrix} + 20\begin{vmatrix} \; \end{vmatrix}$

Theorem 2.2.4 Let E be an $n \times n$ elementary matrix.
a) If E results from multiplying a single row of I_n by a nonzero number k, then $\det(E) = k$.
b) If E results from interchanging two rows of I_n, then $\det(E) = -1$.
c) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

This is a special case of Thm 2.2.3.

#12 Evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

$$\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{array}{ccc} -2 & 4 & 1 \\ 2 & -6 & 0 \\ \hline 0 & -2 & 1 \end{array}$$

$R_3 \to R_3 - 5R_1$

$$\begin{array}{ccc} 5 & -2 & 2 \\ -5 & 15 & 0 \\ \hline 0 & 13 & 2 \end{array}$$

$$= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{17}{2} \end{vmatrix} = -2 \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & \frac{17}{2} \end{vmatrix}$$

$R_3 \to R_3 + \frac{13}{2} R_2$

$$\begin{array}{ccc} 0 & 13 & 2 \\ 0 & -13 & \frac{13}{2} \\ \hline 0 & 0 & \frac{17}{2} \end{array}$$

$$= -2\left(\frac{17}{2}\right) \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -17$$

Note: $\det(A) \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$2 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2\det(A) = 2(ad-bc)$

$2A = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$

$\det(2A) = 4ad - 4bc$

In #16 and 20, evaluate the determinant, given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6.$

#16 $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = 6$

because $R_3 \leftrightarrow R_1$

#20 $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = -12$

$R_3 \to R_3 + 3R_1$

$R_2 \to 2R_2$

Theorem 2.2.5 If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Because an elementary row operation will produce a row of zeros

$$\begin{vmatrix} 2 & 3 & 4 \\ 7 & 9 & 8 \\ -4 & -6 & -8 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 4 \\ 7 & 9 & 8 \\ 0 & 0 & 0 \end{vmatrix} = 0$$